COMS 4281 - Introduction to Quantum Computing Spring 2021

Dirac Notation Worksheet #2

This worksheet is not collected or graded, but meant to help you check your comfort and familiarity with Dirac notation manipulations. The TAs are available during Office Hours to provide assistance if you need any. Answers will be posted after a week.

1 Composite quantum systems

Here are some questions to help you get used to the tensor product.

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. Recall that the tensor product $A \otimes B$ can be given by:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{11}B & A_{12}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}$$

is an $nm \times nm$ dimensional matrix. This is known as the *Kronecker product* which is a way to represent the tensor product with respect to a given basis.

1. Show the explicit vector representations for the standard basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ (i.e. the vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$) using the above definition. What would the explicit vector representation of the standard basis look like for a three qubit system?
2. (a) Give the explicit matrix representations for $I \otimes H$, $H \otimes I$ and $H \otimes H$.
   (b) Let $|x\rangle = |00\rangle$. Compute $(I \otimes H)|x\rangle$, $(H \otimes I)|x\rangle$ and $(H \otimes H)|x\rangle$ directly. Express these results in tensor product notation. (i.e. $\sum_{i,j\in\{0,1\}} c_{ij}|i\rangle|j\rangle$ and some constants $c_{ij} \in \mathbb{R}$)
   (c) If $|\psi_1\rangle|\psi_2\rangle = |\phi_1\rangle|\phi_2\rangle$, then $|\psi_1\rangle$ and $|\psi_2\rangle$ are multiples of $|\phi_1\rangle$ and $|\phi_2\rangle$.

1.1 The EPR pair

Consider the two-qubit state $|\Phi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\Big)$. This is known as the *EPR pair*, after Einstein, Podolsky and Rosen.

1. Suppose we measure the first qubit of $|\Phi\rangle$ (but don't touch the second qubit). Suppose we get outcome $x \in \{0, 1\}$. With what probability does outcome x occur? What is the post-measurement state of $|\Phi\rangle$?
2. Show that the EPR pair is also equal to

$$\frac{1}{\sqrt{2}}\Big(|+\rangle|+\rangle \ + \ |-\rangle|-\rangle\Big) .$$

3. Write out the state $(X \otimes I)|\Phi\rangle$ in terms of the standard basis.
4. Write out the state $(Z \otimes I)|\Phi\rangle$ in terms of the standard basis.
5. Show that $|\Phi\rangle$, $(X \otimes I)|\Phi\rangle$, and $(Z \otimes I)|\Phi\rangle$ are all orthogonal to each other.
6. Show that $|\Phi\rangle = (X \otimes X)|\Phi\rangle = (Z \otimes Z)|\Phi\rangle$.

1.2 Entangling gates

Recall the two-qubit CNOT unitary, which we can define by specifying how it acts on basis states of $\mathbb{C}^2 \otimes \mathbb{C}^2$, namely $|00\rangle, |01\rangle, |10\rangle, |11\rangle$:

$$
\begin{aligned}
CNOT|00\rangle &= |00\rangle \\
CNOT|01\rangle &= |01\rangle \\
CNOT|10\rangle &= |11\rangle \\
CNOT|11\rangle &= |01\rangle \ .
\end{aligned}
$$

1. What is the two-qubit state $CNOT(|+\rangle \otimes |0\rangle)$?
2. From above, show a sequence of operators that generates an EPR pair from the state $|0\rangle \otimes |0\rangle$.
3. Give an explicit matrix representation for the $CNOT$ gate. Can this matrix be represented as a tensor product of single qubit operators?