COMS 4281 - Introduction to Quantum Computing Spring 2021

Dirac Notation Practice Worksheet #1

This worksheet is not collected or graded, but meant to help you check your comfort and familiarity with Dirac notation manipulations. The TAs are available during Office Hours to provide assistance if you need any. Answers will be posted after a week.

1 Bra-ket notation basics

Let $\{|0\rangle, |1\rangle\}$ denote the standard basis for $\mathbb{C}^2$.

Recall from lecture the qubit states $|+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big)$ and $|-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big)$. Also recall the Hadamard gate, which is the unitary $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

1. Show that $H^2 = I$.
2. Show that $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$.
3. Write out the state $|\psi\rangle = \alpha|0\rangle + \beta|+\rangle$ as linear combinations of the standard basis states $|0\rangle, |1\rangle$. In order for $|\psi\rangle$ to be a valid quantum state (i.e. be a unit vector), what are the conditions on α, β?
4. Since $\{|+\rangle, |-\rangle\}$ also forms a basis of $\mathbb{C}^2$ (sometimes called the *diagonal basis*), write $|\psi\rangle$ as a linear combinations of $|+\rangle$ and $|-\rangle$.
5. Write out the state $H|\psi\rangle$ in terms of the standard basis.
6. Write $H|\psi\rangle$ in terms of the diagonal basis.
7. What is the probability of obtaining the $|0\rangle$ and $|1\rangle$ states when measuring $H|\psi\rangle$?

Other important single-qubit unitaries are as follows:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These, along with the identity matrix I, form the famous *Pauli matrices*. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Write out the following states as linear combinations of the standard basis states, i.e. of the form $a|0\rangle + b|1\rangle$:

1. $X|\psi\rangle$
2. $Y|\psi\rangle$
3. $Z|\psi\rangle$

2 Outer products and projections

Recall that $\langle\psi|\theta\rangle$ denotes a scalar, because is an *inner product* between two vectors (i.e., you have a row vector followed by a column vector). Now consider $|\psi\rangle\langle\theta|$, which denotes the *outer product* between the two vectors (i.e., a column vector followed by a row vector). The outer product of two vectors is a matrix.

1. Show that $|0\rangle\langle 0| = \frac{1}{2}(I+Z)$ and $|+\rangle\langle +| = \frac{1}{2}(I+X)$.
2. Show that the operator $|\psi\rangle\langle\psi|$, for any unit vector $|\psi\rangle$, is a projection into a one-dimensional subspace. What is that subspace? What about the operator $\sum_i |\psi_i\rangle\langle\psi_i|$ for an orthonormal set of vectors $\{|\psi_1\rangle, \ldots, |\psi_k\rangle\}$?
3. Let $\{|0\rangle, |1\rangle, \ldots, |n-1\rangle\}$ denote the standard basis for $\mathbb{C}^n$. Show that $I = \sum_{x=0}^{n-1} |x\rangle\langle x|$. This simple identity known as the "completeness relation" can be very useful.
4. Use the completeness relation to show that $|\psi\rangle = \sum_x \langle x|\psi\rangle |x\rangle$ for any vector $|\psi\rangle$. Thus the amplitudes of $|\psi\rangle$, in the standard basis, can be written as inner products of $|\psi\rangle$ with the standard basis vectors.
5. Use the completeness relation to verify the inner product formula

$$\langle\phi|\psi\rangle = \sum_x \langle x|\phi\rangle^* \langle x|\psi\rangle.$$

6. Use the completeness relation to show that for any matrix A we have

$$A = \sum_{x,y=0}^{n-1} \langle x|A|y\rangle |x\rangle\langle y| \ .$$

   In other words the (x,y)-th entry of A in the standard basis is $\langle x|A|y\rangle$. Conclude that $|x\rangle\langle y|$ is a basis for the vector space of operators acting on $\mathbb{C}^n$.

7. Use the completeness relations to verify the familiar matrix-multiplication formula

$$\langle x|AB|y\rangle = \sum_{z=0}^{n-1} \langle x|A|z\rangle\langle z|B|y\rangle.$$