Everything you Need to Know for the First Midterm of Math 317

1. A review of important properties of cross and dot products that are used in the course

   - $\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
   - $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
   - $(c\vec{a}) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right)$
   - $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$
   - $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$
   - $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$
   - $(c\vec{a}) \times \vec{b} = c\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(c\vec{b}\right)$
   - $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$
   - $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = (\vec{c} \cdot \vec{a})\,\vec{b} - \left(\vec{b} \cdot \vec{a}\right)\vec{c}$
   - $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

2. Vector Valued Functions

   - A vector valued function is a function that takes 1 or more variables as an input and outputs a vector in real space (real space is $\mathbb{R}^2$ or $\mathbb{R}^3$ for example).
   - Vector valued functions have the form $\vec{r} = \langle x, y, z\rangle = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
   - We often use parametric representations for the components of $\vec{r}$ using 1 or more parameters (ei. $\vec{r} = \langle x(t), y(t), z(t)\rangle$).
   - If $\vec{r}$ is a vector-valued function pointing to the position of an object moving through space (we are modelling this object as a flying dot), then it's velocity is $\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t)\rangle$, and its position is $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$.
   - The velocity vector is tangent to the curve at the point $\vec{r}(t)$. We can find a vector of length 1 that points in the direction of $\vec{v}$ by normalizing it:

     $$\widehat{T} = \frac{\vec{v}}{\|\vec{v}\|}$$

     This is called the unit tangent vector.

3. Parametrization of a circle

   - A circle in the plane can be parametrically represented by the vector-valued function $\vec{r} = \langle R\cos\theta + a, R\sin\theta + b\rangle$, where R is the radius of the circle, and (a, b) is the center of the circle.

- The speed of the moving object is given by

$$v = \|\vec{v}\| = \dot{\theta}R,$$

where $\dot{\theta}$ (also sometimes ω) is the angular speed (The rate of change of angle subtended by the arc traced by the moving dot with respect to time).

- The acceleration of the moving dot is given by

$$\vec{a} = \frac{dv}{dt}\widehat{T} + R\dot{\theta}^2\widehat{N} = \frac{dv}{dt}\widehat{T} + \frac{v^2}{R}\widehat{N},$$

where $\widehat{N} = \langle -\cos\theta, -\sin\theta \rangle$, the principal unit normal vector, points towards the circle (more on that later).

4. Derivatives and integrals of vector-valued functions

- The fundamental theorem of calculus also applies to vectors

$$\int_{t_0}^{t_1} \frac{d\vec{r}}{dt} dt = \vec{r}(t_1) - \vec{r}(t_0),$$

- Properties of derivatives of vector-valued functions
  (a) Let $\vec{u} = \vec{u}(t), \vec{v} = \vec{v}(t)$ be in $\mathbb{R}^{\nvDash}$ and
  (b) Let $\phi = \phi(t), f = f(x, y, z)$ be scalar functions. Then
    i. $\frac{d}{dt}\left[\phi(t)\vec{u}(t)\right] = \frac{d\phi}{dt}\vec{u} + \phi\frac{d\vec{u}}{dt}$
    ii. $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \cdot \vec{v} + \vec{u} \cdot \frac{d\vec{v}}{dt}$
    iii. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt}$
    iv. $\frac{d}{dt}\left[f\left(\vec{u}(t)\right)\right] = \nabla f\left(\vec{u}(x)\right)\frac{d\vec{u}}{dt}$
    v. $\frac{d}{dt}\left[\vec{u}\left(\phi(t)\right)\right] = \dot{\vec{u}}\left(\phi(t)\right)\dot{\phi(t)}$
    vi. $\frac{d}{dt}\left[\|\vec{u}(t)\|\right] = \left(\vec{u}(t) \cdot \frac{d\vec{u}}{dt}\right) \ / \ \|\vec{u}(t)\|$

5. Rate of change of distance and speed

- The rate of change of speed of an object is given by

$$\frac{d}{dt}\left[\|\vec{r}\|\right] = \frac{\vec{r} \cdot \vec{v}}{\|\vec{r}\|}$$

- If this quantity is positive, the object is moving away from the origin. If it is negative, then the object is moving towards the origin.
- The rate of change of speed is

$$\frac{d}{dt}\left[\vec{v}(t)\right] = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}$$

- If this quantity is positive, then speed and acceleration are at an acute angle. If it is negative, they are at an obtuse angle. If it is 0, then $\vec{v} \perp \vec{a}$ for all t.

6. Arc-length

   - In general, the arc-length of a curve generated by $\vec{r}(t)$ is

   $$s = \int_0^t ds = \int_0^t \|\vec{r}(t)\| dt$$

   - If we reparametrize our function in terms of polar coordinates, where $x = r\cos t, y = r\sin t, r = r(t)$, and $\theta = \theta(t)$, then ds becomes

   $$ds = \sqrt{(dr)^2 + r^2(d\theta)^2}$$

   - If you solve for arclength and then solve for t as a function s, then you can reparametrize a function $\vec{r}(t)$ as $\vec{r}(t(s))$. This function will have the same speed for all t.

7. The independence of curve geometry to parametrization

   - There are an infinite number of parameterizations for a given curve in space
   - The general form of the equation of a line segment is

   $$\vec{r} = \vec{r}_0 + t\vec{v}, \quad a < t < b$$

   where $\vec{r_0}$ is a vector pointing to a point on the line and $\vec{v}$ is a vector parallel to the line. This mimics the form $y = mx + b$ in 2D.

   - It is helpful to think of this through vector addition using the tip-to-tail method. $\vec{r_0}$ points to the line, and $\vec{r}$ points to the sum of this vector and scalar multiples (more specifically t multiples) of $\vec{v}$.
   - Thinking of this definition, you could replace v with any scalar multiple, $c\vec{v}$, since it would be parallel and get the same line. Likewise, you could replace t with any function of t, say $f(t)$, and you would get the same line. This is provided that you adjust the bounds on t so that the line segment starts and ends in the same place.
   - In general, any curve generated by $\vec{r}(t)$ will also be generated by $\vec{r}(u(t))$, provided that u is continuous, smooth, and follows the same constraints as t. However, this does not mean that $\vec{r}(t)$ and $\vec{r}(u(t))$ have the same physical properties.

8. The binormal and principal unit normal vectors

   - The principal unit normal vector, $\widehat{N}$, is the vector that is perpendicular to the unit tangent vector, $\widehat{T}$, and points inward relative to the curve.
   - $\widehat{N}$ is calculated as

   $$\widehat{N} = \frac{\widehat{T}'}{\|\widehat{T}'\|} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{\|(\vec{v} \times \vec{a}) \times \vec{v}\|}$$

- The binormal vector, $\widehat{B}$, is the vector that is orthogonal to both $\widehat{N}$ and $\widehat{B}$.
- $\widehat{B}$ is calculated as
$$\widehat{B} = \widehat{T} \times \widehat{N} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|}$$
- $\widehat{B}, \widehat{N}$, and $\widehat{T}$ can be thought of as an alternative basis that moves with the object along the curve. The plane that is spanned by $\widehat{T}$ and $\widehat{N}$ and that is orthogonal to $\widehat{B}$ is called the osculating plane (from the latin word for to kiss).
- $\widehat{T}, \widehat{N}$, and $\widehat{B}$ have the property that
  (a) $\widehat{T} = \widehat{N} \times \widehat{B}$
  (b) $\widehat{N} = \widehat{B} \times \widehat{T}$
  (c) $\widehat{B} = \widehat{T} \times \widehat{N}$

9. Torsion and curvature

- Curvature, κ, measures how much a curve "curves" it is calculated as
$$\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \|\widehat{T}'(s)\| = \|\vec{r}\,''(s)\| = \frac{\|\widehat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\vec{a} \cdot \widehat{N}}{\|\vec{v}(t)\|^2}$$
The first equation is usually used the most.
- When a curve is constrained to the plane $z = 0$, that is the xy-plane, the formula for curvature reduces to
$$\kappa = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$
if x and y are both functions of t or
$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$
if you have an explicit function for y in terms of x
- The osculating circle at t is the circle that best fits the curve $\vec{r}(t)$ at t.
- The radius of the osculating circle, ρ, is called the radius of curvature and is calculated as
$$\rho = \frac{1}{\kappa}$$
- The torsion (not to be confused with torque) of a curve, τ, is the "out of plane twist." If you imagine a helix in space, the more coiled the helix is, the greater the torsion will be. It also describes how much the osculating plane will "wobble."
- Torsion is calculated as
$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \frac{d\vec{a}}{dt}}{\|\vec{v} \times \vec{a}\|^2}$$

- If a curve is contained in a plane (that is, its points are co-planar), then the torsion, $\tau = 0$.

10. Normal and tangential components of acceleration

    - At this point, we have developed enough tools to find several forms of the equation
    $$\vec{a} = a_T \widehat{T} + a_N \widehat{N}$$
    - The tangent component of acceleration can be written as
    $$a_T = \vec{a} \cdot \widehat{T} = \frac{\vec{r}\,' \cdot \vec{r}\,''}{\|\vec{r}\,'\|} = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$
    - The normal (or sometimes centripetal in physics) component of acceleration can be written as
    $$a_N = \vec{a} \cdot \widehat{N} = v^2 \kappa = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \sqrt{\|\vec{a}\|^2 - a_T^2} = \kappa \left(\frac{ds}{dt}\right)^2$$

11. MGM problems

    - MGM problems arise when we are given some properties of a curve, but not the vector-valued parametric function generating the curve itself.
    - The way of solving these problems consists of three steps:
        (a) Create a “fake” parametrization generating the same geometric curve
        (b) Find intrinsic geometric information (one or more of $\kappa, \tau, \widehat{T}, \widehat{N}, \widehat{B}, \rho$) using this parametrization
        (c) Combine this information with the information provided to find the intended parametrization for the function
        (d) It is important to take note of the direction of t. If, say, x is decreasing, it may be advantageous to use $x = -t$ for a particular curve.

12. Frenet-Serret formulas

    - These are three formulas that relate $\widehat{T}, \widehat{N}$, and $\widehat{B}$ to each-other. They are as follows:
        (a)
        $$\frac{d\widehat{T}}{dt} = v\kappa\widehat{N}$$
        (b)
        $$\frac{d\widehat{B}}{dt} = -v\tau\widehat{N}$$

(c)

$$\frac{d\widehat{N}}{dt} = v\tau\widehat{B} - v\kappa\widehat{T}$$

13. The fundamental theorem of space curves

   - If $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are two smooth parametric curves that are defined on the same interval, $[a, b]$, have the same speed, $v(t)$, curvature, $\kappa(t)$, and torsion, $\tau(t)$, then $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are geometrically congruent. That is, their curves can be moved so that they line up with one another perfectly.

14. Angular momentum

   - Angular momentum is defined as

     $$\vec{H} = \vec{r} \times (m\vec{v})$$

     where m is the mass of the object traced by $\vec{r}$
   - We define a useful quantity

     $$\vec{h} = \frac{\vec{H}}{m} = \vec{r} \times \vec{v}$$

   - If there is no outside torque acting on the system, that is, all forces are parallel to $\vec{r}$, then angular momentum is conserved, and $\vec{h}$ is constant.
   - If this is the case, then all motion will be confined to a plane. This will be used as part of the setup for Kepler's laws.

15. Polar coordinates

   - The polar coordinate system defines a curve using the distance from the origin as a function of the angle travelled counter-clockwise from the positive x-axis, $r(\theta)$.
   - Recall that to go from rectangular coordinates to polar, use $x = r\cos\theta, y = r\sin\theta$.
   - In vector form, rather than using $\hat{\imath}$ and $\hat{\jmath}$ as an orthonormal basis, we use

     $$\hat{r} = \frac{\vec{r}}{\|\vec{r}\|} = \langle \cos\theta, \sin\theta \rangle \text{ and } \hat{\theta} = \langle -\sin\theta, \cos\theta \rangle$$

   - An easy way of remembering this is to just think of the unit circle. You want $\hat{r}$ to point one unit in the direction of the terminal array, so just define it as you would the coordinates of the unit circle. Notice that the components of $\hat{\theta}$ are the derivative of the components of $\hat{r}$. You can actually define these as $\widehat{T}$ and $-\widehat{N}$ for the typical parametric representation of the unit circle.
   - Let $\theta = \theta(t)$. It is important to know these derivative properties and identities for objects moving in the plane:

(a)

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

(b)

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

(c)

$$\hat{r} \times \hat{\theta} = \hat{k}$$

(d)

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

(e)

$$v = \|\vec{v}\| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

(f)

$$\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

(g) If momentum is conserved, then

$$\vec{h} = r^2\dot{\theta}\hat{k}$$

16. Kepler's Laws

- Kepler's laws are mathematical relationships that describe the orbit of the planets around the sun. They can be proved using Newtonian mechanics and rewritten as formulas. They are as follows:
  i. The planets orbit the sun in elliptical paths, with the sun at one of the foci
  ii. For each planet, in equal time intervals, the areas swept out by the cord from the orbiting body to the sun are equal (equal areas in equal time intervals)
  iii. $T^2 \propto a^3$, where T is the orbital period and a is the semi-major axis of the ellipse.
- The setting for Kepler's laws is that the planets are modelled as moving points described by $\vec{r}$ and with mass m, momentum is conserved so that all of the motion of the planets are restricted to a plane (z=0 in the math), and the force experienced by the planets is given by

$$\vec{F}_g = -G\frac{Mm}{r^2}\hat{r} = m\vec{a}$$

In this equation G is the gravitational constant, empirically measured, M is the mass of the sun, m is the mass of the planet, r is the distance between the planet and the sun, and $\vec{a}$ is the acceleration of the planet. We fix the focus with the sun on it at the origin to simplify calculations.

- Kepler II: If A(a,b) is the area swept out by the planet from a to b, then

$$A(a,b) = \int_{\theta(a)}^{\theta(b)} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_a^b h \; dt = \frac{h}{2}(b-a)$$

  This is a mathematical demonstration that A depends only on the length of the time interval

- Kepler I: the polar function $r(\theta)$ describing the distance from the sun to the planet is given by

$$r = \frac{\ell}{1+\varepsilon\cos\theta}, \quad \text{where } \ell = \frac{h^2}{GM}$$

  This is the equation of an ellipse in polar

- Note that the horizontal leftward shift of the ellipse is given by

$$c = \frac{\varepsilon\ell}{1-\varepsilon^2}$$

  The semi-major axis is

$$a = \frac{\ell}{1-\varepsilon^2}$$

  and the semi-minor axis is

$$b = \frac{\ell}{\sqrt{1-\varepsilon^2}}$$

- The variable ε is the eccentricity of the ellipse and describes the ratio between its semi-major and semi-minor axes. We have that

$$\frac{b}{a} = \sqrt{1-\varepsilon^2} \text{ and } 0 < \varepsilon < 1$$

- Kepler III: If T represents the period of the planet's orbit (how long its year is), then

$$T^2 = \frac{4\pi^2}{GM} a^3$$

  This shows that $T^2 \propto a^3$ and their ratio is equal for all planets