October 1, 2025 9:56 AM

Shortcuts for $\int_{\mathcal{C}} f\,ds$

↳ Avg. of $\bar{f} = \frac{1}{L}\int_{\mathcal{C}} f\,ds \iff \int_{\mathcal{C}} f\,ds = L\bar{f}$

↳ This is easy to find if

i. Both L and $\bar{f}$ are known

(case $f(x,y,z) = x$ or y or z might give a centroid coordinate, $\bar{x}$, $\bar{y}$, or $\bar{z}$

↳ It could be easy if L is easy from geometry.

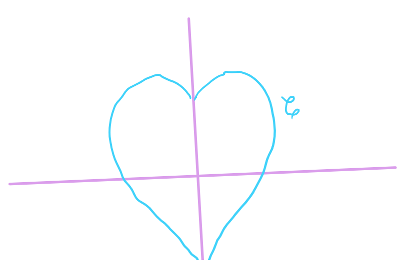
↳ If $f(x,y,z) = k$, a constant, then $\bar{f} = k$, so

$$\int_{\mathcal{C}} (\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} + k)\,ds$$

$$= L(\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} + k)$$)

ii. If $\bar{f} = 0$, no need to know L.

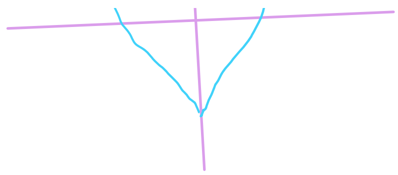
for example:



$\int_{\mathcal{C}} y^2 \sin x = 0$

b/c $f = y^2 \sin x$

is ODD WRT x



is ODD WRT x

$$f(x,y,z) = -f(x,y,z)$$

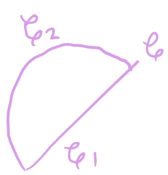
AND curve $\mathcal{C}$ has reflection symmetry

↳ Take $\arccos x = 0$ (A line in $\mathbb{R}^2$ or plane in $\mathbb{R}^3$).

$$\int_{\mathcal{C}} (\pi y^2 \sin x + e^y)\, ds = \underbrace{\cancel{\pi \int_{\mathcal{C}} y^2 \sin x \, ds}}_{0} + \int_{\mathcal{C}} e^y \, ds$$

Now we can simplify 1 peice and calculate the rest

→ Piece wise Curves



if curve $\mathcal{C}$ splits as

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$$

for curves $\mathcal{C}_1$, $\mathcal{C}_2$ (may touch or may not)

then

$$\int_{\mathcal{C}} f\, ds = \int_{\mathcal{C}_1} f\, ds + \int_{\mathcal{C}_2} f\, ds$$

so $L\bar{f} = L_1\bar{f}_1 + L_2\bar{f}_2 \Rightarrow \bar{f} = \dfrac{L_1\bar{f}_1 + L_2\bar{f}_2}{L_1 + L_2}$

If there are n peices, then

$$\bar{f} = \frac{L_1\bar{f}_1 + L_2\bar{f}_2 + \dots + L_n\bar{f}_n}{L_1 + L_2 + \dots + L_n}$$

COM for continuous objects and non-constant $\rho(x,y,z)$

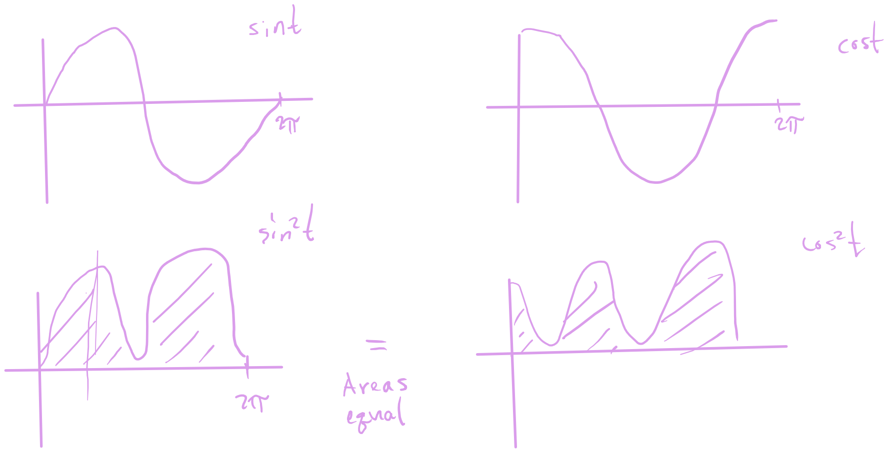
$$\bar{x} = x_{CM} = \frac{\int_{\mathcal{C}} x\, dm}{\int_{\mathcal{C}} dm} = \frac{\int_{\mathcal{C}} x\rho(x,y,z)\, dm}{\int_{\mathcal{C}} \rho(x,y,z)\, dm}$$

$$\bar{x} = x_{CM} = \frac{\cdot}{\int_B dm} = \frac{\cdot}{\int_B \rho(x,y,z)\,dm}$$

Get familiar centroid when $\rho(x,y,z)$ is constant; else this is called a "weighted average" of the x-values.

→ Trig integrals

$$\int_0^{2\pi} \sin^2 t\,dt = \frac{1}{2}(2\pi) = \int_0^{2\pi} \cos^2 t\,dt$$



$$I = \int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \cos^2 t\,dt$$

$$\Rightarrow 2I = I + I = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \frac{1}{2}(2\pi) = \pi$$